

Some Unusual Dimensional Reductions of Gravity: Geometric Potentials, Separation of Variables, and Static - Cosmological Duality

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Abstract

We discuss some problems related to dimensional reductions of gravity theories to two-dimensional and one-dimensional dilaton gravity models. We first consider the most general cylindrical reductions of the four-dimensional gravity and derive the corresponding (1+1)-dimensional dilaton gravity, paying a special attention to the possibility of producing nontrivial cosmological potentials from pure geometric variables (so to speak, from ‘nothing’). Then we discuss further reductions of two-dimensional theories to the dimension one by a general procedure of separating the space and time variables. We illustrate this by the example of the spherically reduced gravity coupled to scalar matter. This procedure is more general than the usual ‘naive’ reduction and apparently more general than the reductions using group theoretical methods. We also explain in more detail the earlier proposed ‘static-cosmological’ duality (SC-duality) and discuss some unusual cosmologies and static states which can be obtained by using the method of separating the space and time variables. This paper is a significantly extended and corrected sum of the recent reports [1] and [2].

1 Introduction

The procedure of dimensional reduction in classical physics is a well known matter but, when one is working with gravity, some subtle points appear, because geometric characteristics of the space-time become dynamical variables. This is most obvious in the Kaluza - Mandel - Klein - Fock reduction (KMKF reduction that usually but unjustly is called KK reduction), in which the metric coefficients become physical fields. This may look less clear in further reductions using cylindrical or spherical symmetries. Then the effective space-time becomes (1+1)-dimensional and some higher-dimensional metric coefficients become dynamical fields, which mix with the original matter fields produced by reductions from higher dimensions.

Low-dimensional models can be obtained by different chains of dimensional reductions from higher-dimensional supergravity or gravity theories (see, e.g., [3] - [9]). For in-

stance, we may consider toroidal compactifications and KMKF reductions from eleven-dimensional theory to a four-dimensional gravity coupled to Abelian gauge fields and scalar fields. In case of spherical or cylindrical symmetry, we can further reduce it to a one-dimensional dilaton gravity coupled to scalar matter fields produced by the reductions. Roughly speaking, such a chain looks like

$$(1 + 10) \rightarrow (1 + D) \rightarrow (1 + 3) \rightarrow (1 + 1) \text{ (spherical or cylindrical).}$$

The two-dimensional theories describe inhomogeneous cosmologies, evolution of black holes, and various types of waves (spherical, cylindrical, and plane waves). Their further reductions give both the standard (or generalized) cosmological models and static states (in particular, static black holes):

$$(1 + 1) \rightarrow (1 + 0) \text{ (cosmological) or } (1 + 1) \rightarrow (0 + 1) \text{ (static).}$$

It is also useful to keep in mind static chains:

$$(1 + 3) \rightarrow (0 + 3) \text{ (general static) } \rightarrow (0 + 2) \text{ (axial) } \rightarrow (0 + 1).$$

We do not consider here some other reductions, like the general axial reduction

$$(1 + 3) \rightarrow (1 + 2) \text{ (axial) } \rightarrow (1 + 1) \text{ (spherical or cylindrical) or } (0 + 2) \text{ (axial).}$$

Note also that it is not necessary to use step-by-step reductions. For instance, the (1+0)-dimensional homogeneous isotropic cosmologies and (0+1)-dimensional static black holes are usually derived by direct symmetry reductions from higher dimensions.

This is quite legitimate, if you are not interested in relations between these reductions and are not trying to immerse them in a more general formulation allowing for their dynamical treatment. In addition, when you have many matter fields, considering first the (1+1)-dimensional dilaton gravity allows us to obtain other interesting solutions, through which the static states, cosmologies, and waves may be inter-related (about relations between various types of solutions see [9]-[11]). Not less important is the fact that lower-dimensional dilaton gravity theories may be regarded as Lagrangian or Hamiltonian systems that are often integrable (in some sense) and thus we may hope to study them in detail and even quantize them in spite of the fact that the general quantum solutions of the higher-dimensional theories can not be constructed. If we make the reductions with due care, we may possibly find important information about solutions of higher-dimensional theories. To succeed in this, one should follow a few important rules which must be used in the process of dimensional reductions.

First, one should not make ‘excessive’ gauge fixings before writing all the equations of motion. For example, the number of independent fields in the reduced theory must be not less than the number of the independent Einstein equations for the Ricci tensor plus the number of the equations for the matter fields. Otherwise, some solutions of the reduced theory will not satisfy (and often do not satisfy) the higher-dimensional equations of motion. Second, by analogy with the usual (‘naive’) reduction, it may be tempting to make all the fields to depend only on one variable (space or time, if we

consider reducing (1+1)-dimensional dilaton gravity). By doing so one can loose some solutions that can be restored with the aid of more general dimensional reductions (e.g., by separating variables). We would like to also emphasize that the concept of dimensional reductions should be understood in a broader sense. An example of a more general dimensional reduction is given in [10], [11]: the solutions of a (1+1)-dimensional integrable model depend on arbitrary moduli functions of one variable; if these functions reduce to constants, we obtain essentially one-dimensional solutions. This reduction may be called a ‘dynamical dimensional reduction’ or a ‘moduli space reduction’. In this example, a class of reduced solutions of the two-dimensional theory consists of those that essentially depend on two space-time variables (say, t and r), which nevertheless should be regarded as ‘one-dimensional solutions’ in a well defined but somewhat unusual sense. Unfortunately, at the moment we can introduce this new dimensional reduction only for explicitly integrable dilaton gravity theories.

To avoid misunderstanding, let us formulate the practical ‘philosophy’ behind our approach to the dimensional reduction of gravity. As distinct from the common tendency to concentrate on geometric and symmetry properties, we follow the Arnowitt - Deser - Misner approach to the treatment of gravity by using Lagrangian and Hamiltonian dynamics with constraints. Thus, the ‘geometric’ variables are treated on the same footing with other dynamical variables and the aim is not only to derive the metric and other geometric properties of the space-time but to construct Lagrangians and Hamiltonians and then to solve the dynamical equations which, eventually, should be quantized. To quantize such a complex nonlinear theory as gravity one should first find some simple explicitly integrable approximation, like the oscillator approximation in the standard QFT. Natural candidates for such ‘gravitational oscillators’ may be static states (e.g., black holes), cosmological models and some simple gravitational waves. One may argue that all these objects are somehow related to the Liouville equation rather than to the oscillator equation [7]-[11].

Although one should not expect that such simplified models can give completely realistic description of gravity, cosmology, or gravitational waves, they may serve as a tool for developing a new intuition, which is so needed for understanding new data on the structure of our Universe. They can also give reasonable first approximations for constructing more realistic solutions as well as some hints of how our main gravitational objects are related physically (at the moment we find only mathematical relations). Using explicitly integrable models one can clearly see a duality between black holes and cosmologies as well as observe that they both are limiting cases of certain gravitational waves. The duality can also be seen in nonintegrable models (e.g., when we use a separation of variables), while the ‘triality’ including some gravitational waves was up to now observed only in integrable theories [10] - [11].

The content of this paper is the following. In Section 2 we summarize the main properties of rather general (1+1)-dimensional dilaton gravity models describing dimensionally reduced (super)gravity theories. Section 3 deals with the dilaton gravity theory obtained by the most general cylindrical reduction of the four-dimensional gravity coupled to scalar fields. This dilaton gravity is more general than usually considered and it was first introduced in [2]. We show that the most general cylindrical reduction

gives an additional (‘geometric’) potential, which drastically changes the properties of the popular integrable SL_2/SO_2 σ -model dilaton gravity. In this paper we also discuss in more detail further dimensional reductions of this generalized ‘cylindrical’ dilaton gravity. In Section 4 we apply the method of ‘dividing and separating’ introduced in [1] to the general ‘spherical’ dilaton gravity and reproduce the concept of duality between the static (in particular, black holes) and cosmological solutions¹. We derive the general constraints that must be satisfied to make the separation possible and briefly outline the construction of several static and cosmological solutions (leaving the complete classification to a future publication). In Conclusion we compare the dimensional reductions considered here to those of papers [9] - [11], in which we observed not only the static-cosmological duality of the exact analytical solutions but also a relation of the static and cosmological states to waves. We also outline the possibility of applying the method of separating to cylindrical and axial static models.

2 (1+1)-Dimensional Dilaton Gravity

It is well known that there exist (1+1)-dimensional dilaton gravity theories coupled to scalar matter fields, which are reliable models for some aspects of high-dimensional black holes, cosmological models, and branes. The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory and in some cases allowed one to find general solution or some special classes of solutions in high-dimensional theories. In this paper, we only discuss reductions of the four-dimensional gravity theory coupled to scalar fields. In fact, after reducing to the dimension (1+1) all the matter fields are essentially equivalent to the scalar ones.

For example, spherically symmetric gravity coupled to Abelian gauge fields and massless scalar matter fields exactly reduces to a (1+1)-dimensional dilaton gravity coupled to scalar fields and can be explicitly solved if the scalar fields are constants independent of coordinates. Such solutions may describe interesting physical objects – spherical static black holes, simplest cosmologies, etc. However, when the scalar matter fields, which presumably play a significant cosmological role, are not constant, few exact analytical solutions of high-dimensional theories are known. Correspondingly, the generic two-dimensional models of dilaton gravity nontrivially coupled to scalar matter are usually not integrable.

Some other important four-dimensional space-times, having symmetries defined by two commuting Killing vectors, may also be described by two-dimensional dilaton gravity. For example, the simplest Einstein - Rosen cylindrical gravitational waves [12] are described by a (1+1)-dimensional dilaton gravity coupled to one scalar field. The simplest stationary axially symmetric pure gravity [13] may be described by a (0+2)-dimensional dilaton gravity coupled to one scalar field (this may be related to the

¹ A relation between the black holes and some cosmologies is known for long time, but it was not clearly formulated and seriously investigated. Similar relations between static states in gravity with matter were demonstrated by some exact solutions of the spherical gravity coupled to matter [7]. The necessity of generalizing the standard naive reductions was explicitly demonstrated in [9].

previous cylindrical case by the analytic continuation of one space variable to imaginary values). Cylindrical waves attracted attention of many researchers for many years (see, e.g., [14], [15]). More recently, similar but more general dilaton gravity models were also obtained in string theory. Some of them may be solved by using modern mathematical methods developed in the soliton theory (see, e.g., [16] - [19]).

Let us briefly remind a fairly general formulation of the (1+1)-dimensional dilaton gravity. The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields ψ_n , which can be obtained by-dimensional reductions of a higher-dimensional spherically symmetric (super)gravity, may usually be (locally) transformed to the following form:

$$L = \sqrt{-g} [U(\varphi)R(g) + V(\varphi, \psi) + W(\varphi)(\nabla\varphi)^2 + \sum_n Z_{nm} \nabla\psi_n \nabla\psi_m]. \quad (1)$$

Here $g_{ij}(x^0, x^1)$ is a generic (1+1)-dimensional metric with signature (-1,1), $g \equiv \det|g_{ij}|$ and $R \equiv R(g)$ is the Ricci curvature of the two-dimensional space-time,

$$ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 0, 1). \quad (2)$$

The effective potentials V and Z_{nm} depend on the dilaton $\varphi(x^0, x^1)$ and on N scalar fields $\psi_n(x^0, x^1)$ ². They may depend on other parameters characterizing the parent higher-dimensional theory, e.g., on charges introduced by solving the equations for the Abelian gauge fields, etc. There are two important simple cases: 1. $Z_{nm}(\varphi, \psi) = \delta_{nm}Z_n(\varphi)$, and 2. constant Z_n , independent of the fields. The dilaton function $U(\varphi)$ is usually monotonic and one can put (at least locally) $U(\varphi) = \varphi$ or $U(\varphi) = \exp(-2\varphi)$, etc. We also may use in Eq. (1) a Weyl transformation to exclude the gradient term for the dilaton, i.e. to make $W \equiv 0$. Under the transformations to this frame (we may call it the Weyl frame) the metric and the potential transform as

$$g_{ij} \rightarrow \tilde{g}_{ij} \equiv w(\varphi)g_{ij}, \quad V \rightarrow \tilde{V} \equiv V/w(\varphi), \quad Z \rightarrow \tilde{Z} \equiv Z, \quad (3)$$

where $w(\varphi)$ is defined by the equation $w'(\varphi)/w(\varphi) = W(\varphi)/U'(\varphi)$.

As we mentioned above, in two-dimensional space-times all matter fields can eventually be reduced to different scalar fields although, for keeping traces of different symmetries, it may be convenient to retain gauge fields, spinor fields, etc. The Lagrangian (1) should be considered as an effective Lagrangian. In general, it is equivalent to the original one on the ‘mass shell’ but the solutions of the original equations may be completely recovered and used to construct the solutions of the higher-dimensional ‘parent’ theory. For a detailed motivation and specific examples see [9], where references to other related papers can be found.

To simplify derivations we will use the equations of motion in the light-cone metric, $ds^2 = -4f(u, v) du dv$ and with $U(\varphi) \equiv \varphi$, $Z_{nm} = \delta_{nm}Z_n$, $W \equiv 0$. By first varying the Lagrangian in generic coordinates and then going to the light-cone ones we obtain the equations of motion

$$\partial_u \partial_v \varphi + f V(\varphi, \psi) = 0, \quad (4)$$

²The potentials Z_{nm} define a negative definite quadratic form.

$$f \partial_i (\partial_i \varphi / f) = \sum Z_n (\partial_i \psi_n)^2, \quad (i = u, v). \quad (5)$$

$$\partial_v (Z_n \partial_u \psi_n) + \partial_u (Z_n \partial_v \psi_n) + f V_{\psi_n}(\varphi, \psi) = \sum_m Z_{m, \psi_n} \partial_u \psi_m \partial_v \psi_m, \quad (6)$$

$$\partial_u \partial_v \ln |f| + f V_\varphi(\varphi, \psi) = \sum Z_{n, \varphi} \partial_u \psi_n \partial_v \psi_n, \quad (7)$$

where $V_\varphi = \partial_\varphi V$, $V_{\psi_n} = \partial_{\psi_n} V$, $Z_{n, \varphi} = \partial_\varphi Z_n$, and $Z_{m, \psi_n} = \partial_{\psi_n} Z_m$. These equations are not independent. Actually, (7) follows from (4) – (6). Alternatively, if (4), (5), (7) are satisfied, one of the equations (6) is also satisfied.

If the Lagrangian (1) was obtained by a consistent reduction of some high-dimensional theory (i.e. not using gauge fixings, which reduce the number of independent equations, and not applying non-invertible transformations to the coordinates or unknown functions), the solutions of these equations can be reinterpreted as special solutions of the parent higher-dimensional equations.

If the scalar fields are constant, $\psi = \psi_0$, these equations can be solved with practically arbitrary potential V that should satisfy only one condition: $V_\psi(\varphi, \psi_0) = 0$, see Eq.(6). The constraints (5) then can be solved because their right-hand sides are identically zero. It is a simple exercise to prove that there exist chiral fields $a(u)$ and $b(v)$ such that $\varphi(u, v) \equiv \varphi(\tau)$ and $f(u, v) \equiv \varphi'(\tau) a'(u) b'(v)$, where $\tau \equiv a(u) + b(v)$ (the primes denote derivatives with respect to the corresponding argument). Using this result it is easy to prove that (4) has the integral $\varphi' + N(\varphi) = M$, where $N(\varphi)$ is defined by the equation $N'(\varphi) = V(\varphi, \psi_0)$ and M is the constant (integral) of motion. The horizon, defined as a zero of the metric $h(\tau) \equiv M - N(\varphi)$, exists because the equation $M = N(\varphi)$ has at least one solution in some interval of values of M . These solutions are actually one-dimensional (‘automatically’ dimensionally reduced) and can be interpreted as black holes (Schwarzschild, Reissner Nordström, etc.) or as cosmological models.

These facts are known for a long time and were derived by many authors using different approaches. A similar solution was obtained in the two-dimensional gravity with torsion [20]. In the standard dilaton gravity, first studied in detail in Ref. [21], the local integral of motion M was constructed in [22] and, by a much simpler derivation, in [7]. The equivalence of the two-dimensional gravity with torsion to the standard dilaton gravity was shown in [23]. The global solutions of the pure dilaton gravity were constructed by many authors (see, e.g., [24]). Systematic studies of matter coupled dilaton gravity models initiated by the CGHS ‘string inspired’ dilaton gravity model [25] resulted in finding more general but simple enough integrable theories (see, e.g., [7]–[10]). A review of different aspects of dilaton gravity and further references can also be found in [26], [27].

With the pure dilaton gravity in mind, it looks, at first sight, natural to introduce the following reduction of the two-dimensional dilaton gravity theories to one-dimensional ones: let φ and ψ depend only on $\tau \equiv a(u) + b(v)$, where τ may be interpreted either as the space or the time variable. Then we obtain both the (0+1)-dimensional theory of static distributions of the scalar matter (including black holes) and (1+0)-dimensional cosmological models. However, analyzing their solutions (see simple examples in [7]) one can find that not all standard Friedmann cosmologies may be obtained in this way

[9], [10]. In view of the symmetry (‘duality’) between the (1+0) and (0+1)-dimensional reductions one may conclude that not all static solutions are obtained by the naive reduction. In other words, this simple (naive) procedure of dimensional reduction is not complete! The same conclusion can be made if we use the space-time variables (t, r) . Before discussing this phenomenon, we consider another simple source of a further incompleteness in the standard processes of reductions.

3 Generalized Cylindrical Reductions

The last remark in the previous section signals that we should apply more care when using dimensional reductions in gravity. To illustrate how more general reductions may emerge we first discuss cylindrically symmetric reductions in the (1+3)-dimensional pure gravity. For acquiring a feeling of connections between the two-dimensional Lagrangian (1) and higher-dimensional theories let us consider the four-dimensional cylindrically symmetric gravity coupled to one scalar field:

$$S_4 = \int d^4x \sqrt{-g_4} [R_4 + V_4(\psi) + Z_4(\psi)(\nabla\psi)^2]. \quad (8)$$

Here the most general cylindrically symmetric metric should be used. It can be derived by applying the general KMKF reduction. The corresponding metric may be written as $(i, j = 0, 1; m, n = 2, 3)$

$$ds_4^2 = (g_{ij} + h_{mn}A_i^m A_j^n)dx^i dx^j + 2A_{im}dx^i dy^m + h_{mn}dy^m dy^n, \quad (9)$$

where all the metric coefficients depend only on the x -coordinates (t, r) while $y^m = (\phi, z)$ are some coordinates on the two-dimensional cylinder (torus).

Usually, in the four-dimensional reduction the coordinate functions A_i^m are supposed to vanish [28], but we will see in a moment that this drastically changes the resulting two-dimensional dilaton gravity theory. To see this, we also suppose that ψ depends only on x and integrate out of Eq.(9) the dependence on y . Extracting the dilaton from the cylinder metric by writing

$$h_{mn} \equiv \varphi \sigma_{mn}, \quad \det(\sigma_{mn}) = 1, \quad (10)$$

and neglecting an inessential numeric factor, we find the two-dimensional Lagrangian (in what follows we will omit the V_4 and Z_4 terms):

$$L = \sqrt{-g} \{ \varphi [R(g) + V_4 + Z_4(\nabla\psi)^2] + \frac{1}{2\varphi}(\nabla\varphi)^2 - \frac{\varphi}{4}\text{tr}(\nabla\sigma\sigma^{-1}\nabla\sigma\sigma^{-1}) - \frac{\varphi^2}{4}\sigma_{mn}F_{ij}^m F^{nij} \}, \quad (11)$$

where $F_{ij}^m \equiv \partial_i A_j^m - \partial_j A_i^m$ ($i, j = 0, 1$). These Abelian gauge fields are not propagating and their contribution is usually neglected. We propose to take them into account by solving their equations of motion and writing the corresponding effective potential. Let us first introduce a very convenient parameterization of the matrix σ_{mn} :

$$\sigma_{22} = e^\eta \cosh \xi, \quad \sigma_{33} = e^{-\eta} \cosh \xi, \quad \sigma_{23} = \sigma_{32} = \sinh \xi. \quad (12)$$

After simple derivations (see, e.g., [7], [9]) we exclude the gauge fields and find the effective potential

$$V_{\text{eff}} = -\frac{1}{2\varphi^2} \sum_{mn} Q_m (\sigma^{-1})_{mn} Q_n = -\frac{\cosh \xi}{2\varphi^2} [Q_1^2 e^{-\eta} - 2Q_1 Q_2 \tanh \xi + Q_2^2 e^{\eta}], \quad (13)$$

where Q_m are arbitrary constants having pure geometric origin, although they look like charges of the Abelian gauge fields F_{ij}^m . Expressing the trace in the Lagrangian (11) in terms of the variables ξ and η , we derive the Lagrangian in our standard form (1):

$$L = \sqrt{-g} \{ \varphi R(g) + \frac{1}{2\varphi} (\nabla \varphi)^2 + V_{\text{eff}} - \frac{\varphi}{2} [(\nabla \xi)^2 + (\cosh \xi)^2 (\nabla \eta)^2] \}. \quad (14)$$

This representation is convenient for writing the equations of motion (5)-(7), for further reductions to dimensions $(1+0)$, and $(0+1)$, and for analyzing special cases (such as $Q_1 Q_2 = 0$, $\xi \eta = 0$). This form is also closer to the original Einstein and Rosen model, which can be obtained by putting $Q_1 = Q_2 = 0$ and $\xi = 0$. It is also more convenient for analyzing the physical meaning of the solutions.

The equations of motion (6) for the Lagrangian (14) are

$$2\varphi \partial_u \partial_v \xi + [\partial_u \varphi \partial_v \xi + (\partial_u \Leftrightarrow \partial_v)] - 2f \partial_\xi V_{\text{eff}} - \varphi \sinh 2\xi \partial_u \eta \partial_v \eta = 0, \quad (15)$$

$$2\varphi \partial_u \partial_v \eta + [\partial_u \varphi \partial_v \eta + 2\varphi \tanh \xi \partial_u \xi \partial_v \eta + (\partial_u \Leftrightarrow \partial_v)] - 2f (\cosh \xi)^{-2} \partial_\eta V_{\text{eff}} = 0. \quad (16)$$

If $\partial_\xi V_{\text{eff}} = 0$ and $\partial_\eta V_{\text{eff}} = 0$, these equations have solutions with constant η and ξ ('scalar vacuum'). However, for $Q_1 Q_2 \neq 0$ we find that the constant solution of the equations $\partial_\xi V_{\text{eff}} = 0$, $\partial_\eta V_{\text{eff}} = 0$ does not exist because ξ should be infinite:

$$\exp 2\eta = Q_1^2 / Q_2^2; \quad \tanh \xi = \text{sgn}(Q_1 Q_2), \quad \text{i.e.} \quad \xi = \pm \infty.$$

If $Q_1 Q_2 = 0$, $Q_1^2 + Q_2^2 \neq 0$, there exists the constant solution, $\xi \equiv 0$, of (15) while $\partial_\eta V_{\text{eff}} \neq 0$ and thus there is no constant solution of (16). We conclude that both ξ and η can be constant if and only if $Q_1 = Q_2 = 0$. This agrees with the fact that the flat symmetry (as well as the generalized spherical symmetry) does not allow for the existence of 'geometric gauge fields' A_i^m (see next Section). In a more general approach, the generalized spherically symmetric configurations should be treated as almost spherical axially symmetric solutions, for which these gauge fields do not vanish. In this sense, the standard spherical solutions are qualitatively different from the 'almost spherical' ones³.

When the potential V_{eff} identically vanishes, Eqs.(15), (16) as well as Eq.(4) drastically simplify and we get the Einstein-Rosen equations for $\xi \equiv 0$. Otherwise we have a nontrivially integrable system of nonlinear equations belonging to the type considered in [16] - [19]. With nonvanishing Q_1 and/or Q_2 , even the further reduced (one-dimensional) equations are nontrivial and it is not quite clear whether they are integrable or not.

³ Our approach can be applied *mutatis mutandis* to considering such solutions of the static axially symmetric theory described by the Ernst equations [13].

Indeed, let us consider the naive reduction of our theory following simple prescriptions of [7]-[10]. Then the fields and metric coefficients depend on one variable $\tau = a(u) + b(v)$ and the effective Lagrangian may be written simply as:

$$L = -\frac{1}{l}[\dot{F}\dot{\varphi} + W\dot{\varphi}^2 - \frac{1}{2}\varphi(\dot{\xi}^2 + \dot{\eta}^2 \cosh^2 \xi)] + lfV_{\text{eff}}. \quad (17)$$

Here $F = \ln |f|$ and $l = l(\tau)$ is the Lagrange multiplier, which can be expressed in terms of the metric g_{ij} (but we do not need this expression here). The equations of motion can be obtained directly from the two-dimensional equations or from this one-dimensional Lagrangian. Before writing the equations, we absorb φ into $1/l$ and define $\phi \equiv \ln |\varphi|$. Then the Lagrangian with the new Lagrange multiplier is

$$L = -\frac{1}{l}[\dot{F}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 \cosh^2 \xi)] + l\varepsilon e^{F+\phi}V_{\text{eff}}, \quad (18)$$

where ε is the sign of f . After writing the equations of motion we can also get rid of l by redefining the evolution parameter τ . Note also that we usually write the equations in terms of the Weyl transformed metric and potential, which are in our case

$$\tilde{F} \equiv F + \frac{1}{2}\phi, \quad \tilde{V}_{\text{eff}} \equiv e^{-\frac{1}{2}\phi}V_{\text{eff}}.$$

With arbitrary charges Q_1 and Q_2 , the reduced theory is probably not integrable. Even when $Q_1Q_2 = 0$, the equations are too complicated to check their integrability. However, in this case the solution with $\xi \equiv 0$ can be explicitly found. Indeed, in this case we have just one term in the potential and the theory can be reduced to the N-Liouville integrable model, which can be explicitly solved [8]-[10]. As was shown in [8]-[10], the static solutions may have up to two horizons. A detailed derivations of this solution will be presented elsewhere.

Now consider the well known integrable case $V_{\text{eff}} \equiv 0$. To solve the one-dimensional equations we do not need to use the inverse scattering method or other advanced theories. The equations of motion can be directly integrated because they reduce to the following simple first-order equations (integrals of motion):

$$2\dot{F}\dot{\phi} + \dot{\phi}^2 = \dot{\xi}^2 + \dot{\eta}^2 \cosh^2 \xi; \quad \dot{\eta} \cosh^2 \xi = c_0; \quad \dot{\xi}^2 = c_1^2 - c_0^2 / \cosh^2 \xi, \quad (19)$$

and, of course, $\dot{\phi} = c_\phi$, $\dot{F} = c_F$ (obviously, $2c_\phi c_F + c_\phi^2 = c_1^2$). Thus we find

$$\xi = \ln\{e^{-T_0}H_\epsilon(T) + [1 + e^{-2T_0}H_\epsilon^2(T)]^{\frac{1}{2}}\}, \quad (20)$$

where ϵ is the sign of $(c_0^2/c_1^2 - 1)$ and we define

$$H_- = \sinh T, \quad H_+ = \cosh T, \quad T \equiv c_1(\tau - \tau_0), \quad T_0 = -\frac{1}{2}\ln|1 - c_0^2/c_1^2|.$$

By simple integrations we then find η :

$$\eta = \eta_0 + \frac{1}{2}\ln[1 + e^{-2(T+T_1)}] - \frac{1}{2}\ln[1 + e^{-2(T-T_1)}], \quad (21)$$

where $|c_0 - c_1|/|c_0 + c_1| \equiv e^{-2T_1}$ and η_0 is a constant.

The expressions for ξ , η , ϕ , and F have no singularities in the interval $-\infty < T < +\infty$. A horizon can appear only when $F \rightarrow -\infty$ while the other three functions are finite. However, ϕ and ξ are infinite for $T \rightarrow \infty$ and thus we have no horizon. On the other hand, as we mentioned above, the solutions of the integrable model corresponding to $Q_1 Q_2 = 0$ may have two horizons. This means that the presence of the potential V_{eff} drastically changes the most important properties of the theory.

In summary of this section, we stress once more that the two-dimensional theories (11) (and the closely related static axial reductions) with vanishing gauge fields were extensively used in cosmology (see, e.g., [29], [30]) and they are integrable with the aid of modern mathematical technique (see, e.g., [19]). However, the effective potential of the geometric gauge fields most probably destroys the integrability, even if we further reduce the theory to one dimension. Nevertheless, the emergence of the potential (13), which under certain circumstances can imitate effects of the cosmological constant, may be of significant interest for the present-day cosmology.

4 Reducing by separating

The spherical reduction apparently does not allow for appearance of the geometric gauge fields⁴ described in the previous section. Correspondingly, the general spherically symmetric metric can be written in a simpler form:

$$ds_4^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega^2(\theta, \phi) - e^{2\gamma} dt^2 + 2e^{2\delta} dr dt, \quad (22)$$

where $\alpha, \beta, \gamma, \delta$ depend on (t, r) and $d\Omega^2(\theta, \phi)$ is the metric on the 2-dimensional sphere $S^{(2)}$. Substituting this into the action (8) and integrating over the variables θ, ϕ we find the reduced action⁵ with the Lagrangian (1), where

$$U \Rightarrow e^{2\beta}, \quad V \Rightarrow 2 + e^{2\beta} V_4, \quad W(\nabla\phi)^2 \Rightarrow 2e^{2\beta}(\nabla\beta)^2, \quad Z_{mn} \Rightarrow Z_4(\psi)e^{2\beta}, \quad (23)$$

and the 2-dimensional metric is given by $e^{2\alpha}, e^{2\gamma}, e^{2\delta}$ (see, e.g., [9]). Actually, the effective two-dimensional Lagrangian also contains total derivatives that may be important in some problems but we will not discuss them here.

The equations of motion for this effective action can easily be derived and they coincide with Eqs.(4) - (7) if we pass to the light cone coordinates. It is not difficult to see (in fact, it is almost evident) that these equations of motion are identical to the Einstein equations (see, e.g., [4]) To simplify the equations, we write them in the limit of the diagonal metric (formally, one may take the limit $\delta \rightarrow -\infty$). Varying the action in α, β, γ and neglecting the δ - terms we obtain the Einstein equations for the diagonal

⁴A very careful discussion of the spherically symmetric space-times and of more general space-times, having subspaces of maximal symmetry, may be found in [31] (see also [32], [33]).

⁵This derivation can easily be generalized to any dimension and any number of the scalar fields with more complex coupling potentials. One can similarly treat the pseudospherical and flat symmetries as well as any symmetry given by two Killing vectors. Here $e^{2\beta}$ is the spherical dilaton denoted by φ in the cylindrical case considered above.

components of the Einstein tensor. Varying the action in ψ we find the equation for ψ . Finally, by varying in δ we find one more equation corresponding to the non diagonal component of the Einstein tensor; it is not a consequence of other equations and is a combination of the two constraints (5).

The simplest way to write all necessary equations is to write the 2-dimensional effective action in the coordinates (22). First making variations in δ we find (in the limit $\delta \rightarrow -\infty$) the constraint

$$\dot{\beta}' + \dot{\beta}\beta' - \dot{\beta}\gamma' - \dot{\alpha}\beta' = \frac{1}{2}Z_4\dot{\psi}\psi', \quad (24)$$

where $\dot{\psi} \equiv \partial_t\psi$, $\psi' \equiv \partial_r\psi$, etc. The other equations can be derived (in the diagonal limit) from the effective Lagrangian

$$L_{\text{eff}} = V_{\text{eff}} + L_t + L_r, \quad (25)$$

where we omitted the δ -dependence and total derivative terms. The sum of the ‘ r -Lagrangian’

$$L_r = e^{-\alpha+2\beta+\gamma}(2\beta'^2 + 2\beta'\gamma' + Z_4\psi'^2), \quad (26)$$

with the ‘ t -Lagrangian’

$$L_t = -e^{\alpha+2\beta-\gamma}(2\dot{\beta}^2 + 2\dot{\beta}\dot{\gamma} + Z_4\dot{\psi}^2), \quad (27)$$

as well as the constraint (24) are invariant under the substitution $\partial_r \Leftrightarrow i\partial_t$ and $\alpha \Leftrightarrow \gamma$. This means that the equation of motion are invariant under this transformation, as the effective potential⁶,

$$V_{\text{eff}} = V_4 e^{\alpha+2\beta+\gamma} + 2k e^{\alpha+\gamma}, \quad k = 0, \pm 1, \quad (28)$$

is naturally invariant. At first sight, this invariance may look trivial but one should recall that in higher dimensions there is no complete symmetry between space and time. Thus the simple relation between static and cosmological solutions suggested by this symmetry may give some new insight into both classes of objects. Even apart from any physical interpretation, this symmetry allows us to economize writing equations and it is extremely useful in considering separation of variables outlined below. In particular, these transformations allow us to derive cosmological solutions corresponding to static (black hole) solutions and vice versa. Although this is a special case of the formulated duality relation we call it ‘static-cosmological’ (SC) duality.

To illustrate how the separation of the variables looks like we write the three remaining equations (in addition to Eq. (23)):

$$[2e^{-2\alpha}(\beta'' + 2\beta'^2 - \beta'\alpha' + \beta'\gamma')] - [\alpha \Leftrightarrow \gamma, \partial_r \Rightarrow \partial_t] = V_{\text{eff}} e^{-\alpha-2\beta-\gamma}, \quad (29)$$

$$[2e^{-2\alpha}(\beta'' + \beta'^2 - \beta'\alpha' - \beta'\gamma')] + [\alpha \Leftrightarrow \gamma, \partial_r \Rightarrow \partial_t] = Z_4 E_+, \quad (30)$$

⁶ Here, in addition to the case of the spherical symmetry ($k = 1$) we include the cases of pseudo-spherical ($k = -1$) and flat ($k = 0$) symmetries.

where we denote

$$E_{\pm} \equiv e^{-2\alpha}\dot{\psi}^2 \pm e^{-2\gamma}\dot{\psi}^2. \quad (31)$$

The third equation has a similar structure

$$[e^{-2\alpha}(\gamma'' + \gamma'^2 - \gamma'\alpha' - \beta'^2) - [\alpha \Leftrightarrow \gamma, \partial_r \Rightarrow \partial_t]] = -ke^{-2\beta} + \frac{1}{2}Z_4E_-, \quad (32)$$

We see that the equations are duality invariant. In practical procedures of the separation it may be convenient to also use an additional (dependent) equation for ψ and work with some linear combinations of the written equations.

To make a separation of the space and the time variables possible we should try to write all the equations in the form

$$\sum_{n=1}^N T_n(t)R_n(r) = 0, \quad (33)$$

where T_n depends only on functions (and their derivatives) of the time variable, while R_n depends only on space functions. Then, dividing by one of the functions and differentiating w.r.t. r or t we finally find equations for functions of one variable depending on constants, which functionally depend on functions of the other variable⁷. For $N = 2$ this is obvious: $T_1/T_2 = R_1/R_2 = C$. For $N = 3$ we may write, for instance,

$$(T_1/T_3)(R_1/R_3) + (T_2/T_3)(R_2/R_3) + 1 = 0$$

and then differentiate this equation w.r.t. r or t , thus reducing the equation to the $N = 2$ case with a new arbitrary constant appearing due to differentiations.

It is evident that, to write the equations in the form (33), we should make some *Ansatz* allowing us to write all the terms as products of functions of one variable. It is clear that to separate the variables r and t in the metric we should require that

$$\alpha = \alpha_0(t) + \alpha_1(r), \quad \beta = \beta_0(t) + \beta_1(r), \quad \gamma = \gamma_0(t) + \gamma_1(r), \quad (34)$$

Then, the potentials V_4 and Z_4 must be either constant or have the necessary multiplicative form. Depending on the analytic form of the potentials, this is possible in two principal cases

$$\psi = \psi_0(t) + \psi_1(r), \quad (\text{a}) \quad \text{or} \quad \psi = \psi_0(t)\psi_1(r), \quad (\text{b}). \quad (35)$$

Here we will not try to find and classify all possible cases of separation and mention only typical ones. If $\dot{\psi}\psi' = 0$, a separation is possible for generic potentials. If $\dot{\psi}\psi' \neq 0$, there are three obvious classes of the potentials that allow the separation: 1. constant potentials V_4 and Z_4 ; 2. exponential $V_4(\psi)$ and $Z_4(\psi)$ (with the *Ansatz* (35a); 3. power dependent $V_4(\psi)$ and $Z_4(\psi)$ (with the *Ansatz* (35b). Note that the case of the constant Z_4 and exponential V_4 is often met in dimensional reductions of gravity and supergravity.

⁷This is one of several possible approaches to solving these equations. We may call it ‘dividing and differentiating’ or simply ‘dd-procedure’.

In four-dimensional theories obtained by the chains of dimensional reductions discussed in Introduction, the potentials are exponentials of linear sums of the ψ -fields. A rather general theory, slightly more general than (8),

$$S_4 = \int d^4x \sqrt{-g_4} [R_4 + V_4(\psi) + \sum_{n=1}^N Z_4^{(n)}(\psi)(\nabla\psi_n)^2], \quad (36)$$

depends on N scalar matter fields ψ_n . The potential V_4 is a sum of linear exponentials of the fields ψ_n ,

$$V_4 = \sum_{k=1}^K g_k \exp \left[\sum_{m=1}^N \psi_m a_{km} \right], \quad (37)$$

and $Z_4^{(n)}$ are either constants or simple linear exponentials of some of the fields ψ (see, e.g., [30] and [9])⁸. With the additive *Ansatz* for ψ ,

$$\psi_n = \psi_{0n}(t) + \psi_{1n}(r), \quad (38)$$

the separation is also possible for the potential and Z terms. To simplify the presentation we discuss here the case of one scalar field with the additive *Ansatz* (the generalization to N fields with the same *Ansatz* is not difficult).

Inserting *Ansatzes* (34) - (35) into the equations (29), (30), (32) we can find further conditions for the separation (when all the equations can be rewritten in the form of Eq.(33)). To simplify the discussion we choose⁹: $\alpha_1 = 0$ and $\gamma_0 = 0$. Note that this does not significantly restrict our local considerations because this is equivalent to changing the coordinates (t, r) to (\bar{t}, \bar{r}) :

$$\bar{t} = \int dt e^{\gamma_0(t)}, \quad \bar{r} = \int dr e^{\alpha_1(r)}.$$

Then, analyzing (29) - (32), we see that all the terms, except $ke^{2\beta}$ (for $k \neq 0!$), will have the form $T_n(t)R_n(r)$ after multiplying the equations by $e^{2\alpha_0}e^{2\gamma_1}$. By applying the dd-procedure it is easy to prove that $ke^{2\beta}$ can be presented in the required form if at least one of the two (dual) conditions,

$$\text{I. } \dot{\alpha}_0 = \dot{\beta}_0 \quad \text{or} \quad \text{II. } \beta_1' = \gamma_1', \quad (39)$$

is satisfied for $k \neq 0$. When $k = 0$, this restriction is unnecessary.

The second strong restriction on the separated metric functions is given by the constraint (24). Let us first consider the case $\dot{\psi}\psi' = 0$. Then it is not difficult to prove that its solutions are:

$$1. \dot{\alpha}_0 = \dot{\beta}_0 = 0; \quad 2. \beta_1' = \gamma_1' = 0; \quad (40)$$

$$3. \dot{\alpha}_0 = 0, \quad \beta_1' = \gamma_1'; \quad 4. \gamma_1' = 0, \quad \dot{\alpha}_0 = \dot{\beta}_0; \quad 5. \dot{\beta}_0 = \beta_1' = 0. \quad (41)$$

⁸The cylindrical theory and the static axial theory also belong to this class.

⁹Sometimes, this may be inconvenient. For example, for the standard form of the Schwarzschild solution $2\alpha_1 = -\ln(1 - r_0/r)$.

When no one of this conditions is satisfied, (24) can be written as

$$\gamma_1'/\beta_1' + \dot{\alpha}_0/\dot{\beta}_0 = 1, \quad (42)$$

and the solution of this equation is obviously $(-\infty < C < +\infty)$

$$6. \gamma_1' = (1 - C)\beta_1', \quad \dot{\alpha}_0 = C\dot{\beta}_0. \quad (43)$$

The conditions 2 and 4 are dual to the conditions 1 and 3, respectively. The conditions 5 and 6 are obviously self-dual, the conditions 3 and 4 follow from (43) for $C = 0$ and $C = 1$, respectively. If we take into account the condition (39), we find that for $k \neq 0$ there are four basic configurations: A. $\dot{\alpha}_0 = \dot{\beta}_0 = 0$ or the dual; B. $\dot{\alpha}_0 = \dot{\beta}_0$, $\gamma_1' = 0$ or the dual. The A-configuration corresponds to naive static or cosmological reductions. The B-configuration cannot be obtained by naive reductions. With $k = 0$, we simply use the conditions (40) - (43).

One can similarly treat the case $\dot{\psi}\psi' \neq 0$. To simplify the discussion let us suppose that $Z_4 = -2$. Applying the dd-procedure to the constraint (24) written as

$$\frac{\dot{\alpha}_0}{\dot{\psi}_0} \frac{\beta_1'}{\psi_1'} + \frac{\dot{\beta}_0}{\dot{\psi}_0} \frac{\gamma_1' - \beta_1'}{\psi_1'} - 1 = 0, \quad (44)$$

we can prove that its solutions are

$$\dot{\alpha}_0 = (1 - C)\dot{\beta}_0, \quad \dot{\psi}_0 = C_1\dot{\beta}_0, \quad \psi_1' = C_1^{-1}(\gamma_1' - C\beta_1'), \quad (45)$$

$$\dot{\beta}_0 = 0, \quad \dot{\psi}_0 = C_1^{-1}\dot{\alpha}_0, \quad \psi_1' = C_1\beta_1', \quad (46)$$

and two solutions that are dual to these.

Now, if the potentials V_4 and Z_4 are separable and the constraint (24) as well as one of the constraints (39) (for $k \neq 0$) are satisfied, all the equations of motion can be written in the separated form (33). If there are several scalar fields ψ_n we should add the equations obtained by varying the Lagrangian also w.r.t. these fields. Following this way, we can obtain all the standard black holes and cosmologies and, in addition, other spherically symmetric static and cosmological solutions related by the SC-duality. It is important to emphasize that, when we have many scalar matter fields and rather complex potentials, the equations of motion are, in general, not integrable. Note also that the distinction between the cosmological and static solutions is not trivial because general reduced solutions depend on both r and t . Nevertheless, we should regard the procedure of separation as a dimensional reduction. The meaning of this can be clarified by the following examples.

Let us first choose condition 2 of (40), which says that β_1 and γ_1 are constant¹⁰ (without loss of generality we may suppose that they vanish). Then the equations of motion (29) - (32) for the remaining dynamical functions α_0 , β_0 , ψ_0 ,

$$(\ddot{\beta}_0 + 2\dot{\beta}_0^2 + \dot{\beta}_0\dot{\alpha}_0) + ke^{-2\beta_0} = -\frac{1}{2}V_4(\psi_0), \quad (47)$$

¹⁰A cosmological model with this metric was studied in [34], [35].

$$(\ddot{\beta}_0 + \dot{\beta}_0^2 - \dot{\beta}_0 \dot{\alpha}_0) = \frac{1}{2} Z_4 \dot{\psi}_0^2, \quad (48)$$

$$(\ddot{\alpha}_0 + \dot{\alpha}_0^2 - \dot{\beta}_0^2) - k e^{-2\beta_0} = \frac{1}{2} Z_4 \dot{\psi}_0^2, \quad (49)$$

define a cosmological model. From (47) and (48) we derive the integral of motion, which is the total (gravity plus matter) energy of the system

$$\dot{\beta}_0^2 + 2\dot{\beta}_0 \dot{\alpha}_0 + k e^{-2\beta_0} + \frac{1}{2} V_4(\psi_0) + \frac{1}{2} Z_4 \dot{\psi}_0^2 = 0. \quad (50)$$

Thus we find the complete system of equations for α_0 , β_0 , ψ_0 (recall that we take the coordinates in which $\alpha_1 = \gamma_0 = 0$). Of course, one can see that we recovered the naive cosmological reduction. This cosmology does not coincide with the standard FRW cosmology, which can be obtained by using the conditions $\gamma_1' = 0$, $\dot{\alpha}_0 = \dot{\beta}_0$, and $\psi_0^{prime} = 0$. The naive cosmology is unisotropic because $R_{11}^{(3)} = 0$ while $R_{22}^{(3)} = k$, where $R_{ij}^{(3)}$ is the Ricci tensor of the 3-dimensional subspace (r, θ, ϕ) . For the naive cosmology, the Ricci curvature of the 3-space is simply $R^{(3)} = 2k e^{-2\beta_0(t)}$.

The naive static reduction may be reproduced simply by using our SC-duality. The Schwarzschild black hole then can be obtained if we take $V_4 = 0$ and $Z_4 = 0$. Otherwise we have a static state of gravitating scalar matter. For the generic functions $V_4(\psi)$ and $Z_4(\psi)$ the equations of motion for both dual theories are not integrable but, if the static theory has horizons (this does not contradict to the ‘no hair’ theorem, if V_4 depends on ψ , see [7]-[10]), we may construct analytic perturbation theory near each horizon (see [36]). It would be interesting to construct a ‘dual’ perturbation theory for the cosmological solutions. Note that the perturbation theory can be applied not only to the naive reductions.

With less restrictive *Ansatzes* we may construct other cosmological models and static configurations that are dual to them. For example, with the condition (43), we find in the equations of motion the terms depending both on r and on t . Thus the more general procedure of separation should be applied. The interpretation of the solutions as cosmological or static requires more care and will be discussed in a separate publication, where further examples will also be presented. Constructing the one-dimensional Lagrangians producing the reduced equations of motion requires more care also. Especially, we should not completely fix the gauge. Even in the naive reduction (47) - (50) we must avoid choosing the obvious gauge $\gamma_0 = 0$ because e^{γ_0} plays the role of the Lagrange multiplier $l(\tau)$ in the one-dimensional Lagrangian (17).

It is interesting that there may exist some ‘intermediate’ cases that are more symmetric under the duality transformation being neither static nor cosmological. It would be premature to call them ‘self - dual’ before a detailed study of them will be undertaken.

Here we considered the separating of variables approach for the spherically reduced gravity. With due care, it can be applied to the generalized cylindrical theory (14). It is of interest to apply separating to reducing static axial theory with KMKF potentials. This may allow us to describe essentially generalized perturbed spherical states that may be considered as more realistic models of black holes.

5 Conclusion

In a separate publication I will present a complete list of all mentioned reductions and their relation to black holes, cosmologies, and waves (especially, the cylindrical ones). I will also make an attempt to compare the models obtained by the approach of this paper to known cosmological and static solutions derived by other methods and classified by their group theoretical properties. At the moment, relations of our dynamical classification to the group theoretical ones is not clear.

An interesting new topic is dimensional reduction to waves. Here I only mention the wave - like solution obtained in the integrable model of (1+1)-dimensional gravity coupled to N scalar matter fields [9], [10]. The general solution of the model depends on the chiral moduli fields $\xi_n(u)$, $\eta_n(v)$ that move on the surfaces of the spheres $S^{(N)}$. The naive reduction to one-dimensional theories emerges when the moduli fields are constant and equal, $\xi_n = \eta_n$. When they are constant but otherwise arbitrary, we have a new class of reduced solutions that correspond to waves of scalar matter coupled to gravity. Under certain conditions, these waves may be localized in space and time and thus may be regarded as a sort of solitary gravitational waves¹¹. The very origin of these waves signals existence of a close relation between main gravitational objects - black holes, cosmologies and waves. This relation was studied in some detail for static states and cosmologies and so was called the static-cosmological duality. In the integrable models, transitions between static and cosmological states are possible and, moreover, the waves play a significant role in these transitions. This observation, which does not actually require integrability, may open a way to studies of real physical connections between these apparently diverse objects.

In summary, one may identify at least three types of dimensional reduction: the ‘standard’ or ‘naive’ reduction which supposes that functions of two variables depend on one variable only, the reduction by separating the variables, and the reduction in moduli spaces supposing that the moduli functions become constants. In all cases the important problem is to find the Lagrangians and Hamiltonians for the reduced systems. This is not difficult for naive reductions and for simple reductions based on separating. It is not clear how to do this with the last, so to speak, ‘moduli reduction’. In addition, it is not clear how to do such a reduction for not integrable systems.

Finally, we wish to emphasize once more that the ‘geometric potentials’ can emerge in ‘almost spherical’ (perturbed axial) states. It must be not very difficult to apply our considerations to such states and we hope to do this in near future.

¹¹ A special solution of this kind has recently been found in [10] and will be generalized and discussed in more detail in the forthcoming paper [11]. Note that our solitary waves do not seem to have a relation to possible soliton - like states in the theories with the ‘sigma - model’ - like coupling of the scalar fields to gravity, which can be obtained from (14) with $V_{\text{eff}} \equiv 0$ (they are studied in [16] - [19], see also a discussion in [10] as well as a simplified explicitly soluble model of scalar waves in dilaton gravity proposed in [37]).

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